

AD-A052 823

ILLINOIS INST OF TECH CHICAGO

F/G 20/11

UNIFIED ELASTIC PLASTIC CONSTITUTIVE EQUATION AND APPLICATION T--ETC(U)

MAR 78 B BERNSTEIN

DAA629-77-6-0082

UNCLASSIFIED

ARO-14716.3-R-E

NL

1 OF 1  
ADA  
052823

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

11  
11  
11

END  
DATE  
FILMED  
5-78  
DDC

ARO 14716.3-R-E

AD A 052823

AU NO. 1  
DDC FILE COPY

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. JOINT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A Unified Theory of Elasticity and Plasticity and Formulation of Hydraulic Autofrettage Problems		5. TYPE OF REPORT & PERIOD COVERED Final 15 Jan. 1977-14 Dec. 1977
7. AUTHOR(s) Barry Bernstein		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Illinois Institute of Technology Chicago, Illinois 60616		8. CONTRACT OR GRANT NUMBER(s) Grant No. DAAG29-77-G-0082
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12211 Research Triangle Park, NC 27709		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS P-14716-R-E
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		12. REPORT DATE 15 March, 1978
		13. NUMBER OF PAGES 19
		15. SECURITY CLASS. (of this report) unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES The findings in this report are not to be construed as an official Department of the Army position, unless so designated by other authorized documents.		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Plasticity, Yield, Autofrettage, Generalized Plane Strain, Lipshitz Condition, Prandtl-Reuss Equations		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Work is reported on the development of a set of unified rate type constitutive equations for elastic and plastic behavior, and of the formulation of generalized plane strain problems of hydraulic autofrettage in terms of these equations, as well as some initial computations in terms of the formulation. The unified equations apply to both loading and unloading. Elasticity is related to uniqueness of solutions of differential equations and occurs when a Lipshitz condition is in force. Plastic yield response occurs during loading		

175400

Block 20. (Continued)

when the Lipshitz condition fails. Failure of the Lipshitz condition corresponds to the von Mises yield condition. During yield, the equations automatically become those of Prandtl-Reuss. During unloading, the stable solution dictates a reentry to an elastic regime. Linear elastic-perfectly plastic behavior can be approximated arbitrarily closely. Extension to strain hardening can be included. The problem of generalized plane strain in hydraulic autofrettage is reduced to a system of ordinary differential equations with initial conditions. A computer program is written for the open end condition in the incompressible case with an instruction which insures a slight perturbation during yield. The computer solution shows that yield does occur during loading and that a return to elastic behavior during unloading occurs.



ACCESSION for	
RTIS	White Section <input checked="" type="checkbox"/>
DDG	Defi Section <input type="checkbox"/>
UNANNOUNCED	<input type="checkbox"/>
JUSTIFICATION.....	
BY.....	
DISTRIBUTION/AVAILABILITY CODES	
Dist.	AVAIL. and/or SPECIAL
A	

⑥ Unified Elastic Plastic Constitutive  
Equation and Application to  
Hydraulic Autofrettage Problems

⑨ Final Report. 15 Jan - 14 Dec 77

⑩ Barry Bernstein

⑪ 15 March 1978

⑫ 21 p.

U.S. Army Research Office  
Grant number ⑮ DAAG29-77-G-0082

⑰ ARO

⑱ 14716.3-R-E

Illinois Institute of Technology  
Chicago, Illinois 60616

DISTRIBUTION STATEMENT A  
Approved for public release;  
Distribution Unlimited

DDC  
RECEIVED  
APR 18 1978  
D

175 400

SP



THE FINDINGS IN THIS REPORT ARE NOT TO BE  
CONSTRUED AS AN OFFICIAL DEPARTMENT OF  
THE ARMY POSITION, UNLESS SO DESIGNATED  
BY OTHER AUTHORIZED DOCUMENTS.

RECEIVED  
APR 18 1978  
D D C

DISTRIBUTION STATEMENT A  
Approved for public release  
Distribution Unlimited

### Statement of Problem Studied

The problem studied is the search for a common logical approach to elastic and plastic behavior. In particular the work involved development of the formulation of a set of rate type constitutive equations which exhibits both elastic response and plastic yield as well as transition between the two types of behavior, both in loading and unloading. The relation to classical linear elastic-perfectly plastic behavior was studied and strain hardening was considered. Formulation of the hydraulic autofrettage problem in generalized plane strain was studied. Plane strain, as well as open and closed end conditions were considered. Computation for the open end condition was begun for the incompressible case.

### Personnel

1. Barry Bernstein, Professor of Mathematics, Illinois Institute of Technology.
2. Arsalan Shokooh, graduate student. Mr. Shokooh completed his Ph.D. degree during the time he was working on this project, although his thesis was essentially complete before he started his work on this project.
3. William F. Drish, graduate student.

### List of Publications and Reports

- Report number 1: Progress Report,  
January 15 to June 30, 1977
- Report number 2: Bernstein, Barry, "A Unified Theory of  
Elasticity and Plasticity", Int. J.  
Engineering Science 15, pp. 645-660 (1977).

Report number 3: Bernstein, Barry, "Unified Constitutive Relations for Elastic and Plastic Behavior", Recent Advances in Engineering Science - Proceedings of the Fourteenth Annual Meeting, Society of Engineering Science, Lehigh University, November 14-17, 1977, pp. 773-778.

Report number 4: Final Report - Including summary of progress up to time of end of project.

Oral Presentation: "Unified Constitutive Relations for Elastic and Plastic Behavior", B. Bernstein, Fourteenth Annual Meeting of Society of Engineering Science, Lehigh University, Bethlehem, Pennsylvania, November 14, 1977.

### Summary of Scientific Work

Introductory Note: We summarize here briefly the work in References [1] and [2], (Reports 2 and 3). In addition, we report in some detail the progress on the problem of hydraulic autofrettage during the duration of the project.

1. Introduction. The purpose of the efforts on this project was to formulate properly a unified set of constitutive equations for elastic and plastic behavior and to study methods of applying the formulation to advantage in solving problems of autofrettage. During the duration of the project, a formulation was brought to completion and published [1]. Methods were advanced for solving the hydraulic autofrettage problem in generalized plane strain as a step toward developing methods of applying the equations. Computations were carried out to test the method.

Although the incompressible plane strain is considered too simple to be of great interest in itself, some study was made of it initially in order to test some aspects of writing computer programs for the equations. Next the cases of open end and closed end conditions were taken up for generalized plane strain.



Using to advantage the unified nature of the equations, we were able through mathematical analysis to reduce these problems to initial value problems for a set of ordinary differential equations, which then can be easily and cheaply solved by an automatic digital computer. A computer program was written and computations were carried out. The computations demonstrated clearly that the method does give an automatic transition to yield during loading and an automatic return to elastic behavior during unloading. This is where the work stood at the time the project ended. Although we report numerical results which show the transition to and from yield, the program written needs further refinement before the numbers obtained are presentable as representing the actual predicted stress distribution. These refinements are straightforward, involving improved integration techniques and better error bounds, and will be carried out in the future. The short duration of the project did not allow time for these refinements. The results are, however, successful in what they were intended to test, as will be presented below.

2. Development of the Unified Equations. The results of the investigations on the unified equations was published in the International Journal of Engineering Science under the title of "A Unified Theory of Elasticity and Plasticity" [1]. A presentation was also made at the fourteenth annual meeting of the Society of Engineering Science under the title of "Unified Constitutive Equations for Elastic and Plastic Behavior" and an extended abstract was published in the proceedings of that meeting [2]. We shall present here a brief summary.

A rate-theory equation was obtained which relates stress-deviator  $s_{ij}$  and strain deviator  $\epsilon_{ij}$ . The equation has the form

$$\dot{s}_{ij} = \frac{\beta}{\psi(\beta)} \dot{\epsilon}_{ij} + \frac{1}{\beta^2} \left[ \frac{1}{\psi'(\beta)} - \frac{\beta}{\psi(\beta)} \right] s_{ij} s_{kl} \dot{\epsilon}_{kl}, \quad (1)$$

where

$$\beta = \sqrt{s_{ij} s_{ij}}$$

and  $\psi$  is an unspecified function. It was shown by construction that it is possible to choose the function  $\psi$  so as to make equation (1) give rise to both elastic and plastic response as well as transition to plastic yield during loading and from plastic yield during unloading with a von Mises yield condition. Indeed, yield corresponds to failure of a Lipschitz condition with its possibility of multiple solutions and elastic behavior results from uniqueness of solutions when the Lipschitz condition holds. During loading, which is characterized by positive shear work rate, yield is highly stable. During unloading (negative shear work rate) yield is unstable. Thus the assumption that the stable solution is followed implies a return to an elastic regime upon the onset of unloading from yield. (This notion is readily incorporated into computer solutions.) During yield, the basic constitutive equations automatically become the Prandtl-Reuss equations. Furthermore it was shown how  $\psi$  could be chosen so that the elastic response approximates to any desired degree a linear elastic behavior. An extension to strain hardening was constructed.

3. The Problem of Hydraulic Autofrettage. It must be understood that the work at this stage is intended to develop new methods of solving autofrettage problems. It is natural, then, first to attack old problems by a new method before launching on new ones in order to test the method. Such has been the intent of the work done so far. We have during the duration of the project brought the work to the point of establishing the feasibility and ease of applying computer techniques to the unified equations. In particular, there are three matters which have been tested: 1) advantages of the unified equations in analyzing a problem in preparation for computation, 2) the transition to yield during loading in a computer solution and 3) the use of the stability criterion to determine when there is persistence of yield and when there is a return to elastic behavior. We feel that these tests have been successfully met.

Work was done on the case of generalized plane strain with different end conditions. Analysis for both the incompressible and compressible cases were performed. However, to date time has permitted only the incompressible case to be brought to the stage of computation. We shall now discuss generalized plane strain.

The (physical) components of displacement  $u_i$  in cylindrical coordinates  $r, \theta, z$  are taken to be

$$u = u_r, \quad v = u_\theta, \quad w = u_z,$$

and for generalized plane strain we assume

$$u = u(r), \quad v = 0, \quad w = Az + B.$$

For the incompressible case, the equation of incompressibility gives



$$\frac{\partial u}{\partial r} + \frac{u}{r} + A = 0 \quad (2)$$

whence

$$u = \frac{C}{r} - \frac{Ar}{2}, \quad (3)$$

where  $C$  is a constant of integration. The stress  $\sigma_{ij}$  and the stress deviator  $s_{ij}$  are related by

$$\sigma_{ij} = -p \delta_{ij} + s_{ij}$$

where

$$p = -\frac{\sigma_{kk}}{3}, \quad s_{ii} = 0.$$

Solutions are then sought for which the off-diagonal components of  $s_{ij}$  vanish, and we write more simply  $s_r$  for  $s_{rr}$ ,  $s_\theta$  for  $s_{\theta\theta}$  and  $s_z$  for  $s_{zz}$ . Equilibrium then gives

$$\frac{-\partial p}{\partial r} + \frac{\partial s_r}{\partial r} + \frac{s_r - s_\theta}{r} = 0 \quad (4)$$

or

$$\frac{\partial \sigma_r}{\partial r} + \frac{s_r - s_\theta}{r} = 0. \quad (5)$$

We took, as in reference [1],  $\psi$  to satisfy

$$\psi'(\beta) = \frac{1}{2\mu} \left( 1 - (\beta^2 / 2k^2)^{2n} \right)^{-\frac{1}{2}}$$

where  $n$  is an integer,  $\mu$  is a constant giving initial shear modulus and  $k$  is a constant giving shear yield stress. The quantities were normalized or non-dimensionalized in such a way as to lead to the same result as taking  $2\mu = 1$  and  $2k^2 = 1$ .

(This can be done by replacing  $s_{ij}$  by  $s_{ij}/k$  and  $u_i$  by  $\mu u_i/k$ .)

We shall then assume that the quantities actually represent their non-dimensionalized equivalents, whence we write

$$\psi' = (1 - \beta^{2n})^{-\frac{1}{2}}$$

and obtain  $\psi$  as in reference [1]. Then equations (1) and (3) give for the incompressible plane strain

$$\begin{aligned}\dot{s}_r &= \frac{-\beta}{\psi} \frac{\dot{C}}{r^2} + \frac{1}{\beta^2} \left[ \sqrt{1 - \beta^{2n}} - \frac{\beta}{\psi} \right] \left[ \frac{-2\dot{C}}{r^2} s_r^2 + s_r s_z \left( \frac{3}{2} \dot{A} - \frac{\dot{C}}{r^2} \right) \right] \\ \dot{s}_z &= \frac{\beta}{\psi} \dot{A} + \frac{1}{\beta^2} \left[ \sqrt{1 - \beta^{2n}} - \frac{\beta}{\psi} \right] \left[ \frac{-2s_r s_z}{r^2} \dot{C} + s_z^2 \left( \frac{3}{2} \dot{A} - \frac{\dot{C}}{r^2} \right) \right].\end{aligned}\quad (6)$$

Let  $a$  be the inner radius and let  $b$  be the outer radius of the cylinder which is to be stressed. For all cases we shall have  $\sigma_r = 0$  at  $r = b$ . Let  $P$  be the hydraulic pressure of the fluid inside the cylinder. Then for all cases we require that  $\sigma_r = -P$  at  $r = a$ . The conditions distinguishing the three cases are respectively the following:

Case I: Plane strain:  $A = 0$ .

Case II: Open end condition:

$$\int_a^b r \sigma_z dr = 0 \quad (7)$$

Case III: Closed end condition:

$$\int_a^b r \sigma_z dr = \frac{a^2 P}{2} \quad (8)$$

We now discuss the cases more specifically:

Case I: Plane strain,  $A = 0$ . In this case we can take  $s_z = 0$  and the problem reduces to solving the simple equation

$$\dot{\beta} = \frac{\sqrt{2}}{r^2} \sqrt{1 - \beta^{2n}} \dot{c} ,$$

with  $s_r = -\beta/\sqrt{2}$  for which we could take  $C$  as a parameter:

$$\frac{d\beta}{dC} = \frac{\sqrt{2}}{r^2} \sqrt{1 - \beta^{2n}} . \quad (9)$$

Equation (9) was solved numerically at a number of  $r$  locations using a Hamming Predictor-Corrector Scheme [3]. The following instruction was added: If  $\beta \geq 1$ , replace  $\beta$  by  $N$ , where  $N$  was taken to be a number slightly less than unity, and  $n$  was given several integer values, 1, 20, 32, 64.  $N$  was given values .99 and .9999. Achievement of yield occurred as expected during loading, and a return to elastic behavior occurred during unloading. Indeed, the only difficulty which was experienced occurred for the physically unimportant small  $n = 1$ , where it was found that for the values of  $N$  chosen, extra accuracy would have to be introduced into the integration scheme in order to get a transition to the proper elastic regime. This phenomenon was ascribed to the relatively slow change in slope near yield for  $n = 1$  and did not occur for the higher values of  $n$ . This observation leads to some interesting mathematical questions, which were not pursued at this stage since they did not affect the cases of interest for which the scheme worked well. Once  $\beta$  is known as a function of  $r$  and  $C$ , we have  $s_r = -\beta/\sqrt{2}$ ,  $s_z = 0$ ,  $s_\theta = \beta/\sqrt{2}$ , and  $P$  can be determined by solving (8) with  $\sigma_r(b) = 0$  and using  $P = -\sigma_r(a)$ . The incompressible plane strain case, then, is basically one dimensional and quite trivial. We go on to the other two cases.



Cases II and III: Open and Closed End Conditions: Note that

$$\sigma_z = \sigma_r + s_z - s_r$$

whence we can write for the open end condition (7)

$$\int_a^b r \sigma_r dr + \int_a^b r(s_z - s_r)dr = 0. \quad (10)$$

For the closed end condition (8), we get similarly

$$\int_a^b r \sigma_r dr + \int_a^b r(s_z - s_r)dr = \frac{a^2 P}{2}. \quad (11)$$

Integration by parts and the conditions at the inner and the outer radius give

$$\int_a^b r \sigma_r dr = \frac{Pa^2}{2} - \int_a^b \frac{r^2}{2} \frac{\partial \sigma_r}{\partial r} dr. \quad (12)$$

Substitution of  $\partial \sigma_r / \partial r$  from (5) into (12), replacement of  $s_\theta$  by  $-s_r - s$ , and substitution of the result into (10) and into (11) gives

$$\int_a^b r s_z dr = - \frac{Pa^2}{3} \quad (13)$$

for the open end condition and

$$\int_a^b r s_z dr = 0$$

for the closed end condition respectively.

Integration of (5) and use of the inner and outer radius conditions gives

$$-P = \int_a^b \frac{2s_r + s_z}{r} dr. \quad (14)$$

For the open end condition, we obtain by differentiating (13) and (14) and substituting (6)

$$\begin{aligned} \frac{-a\dot{2}\dot{P}}{3} &= T\dot{A} + L\dot{C} \\ -P &= M\dot{A} + Q\dot{C} \end{aligned} \quad (15)$$

where

$$\begin{aligned} T &= \int_a^b r \left[ \frac{\beta}{\psi} + \frac{3}{2} \frac{s_z^2}{\beta^2} \left( \sqrt{1 - \beta^{2n}} - \frac{\beta}{\psi} \right) \right] dr \\ L &= - \int_a^b \frac{1}{r\beta^2} \left[ \sqrt{1 - \beta^{2n}} - \frac{\beta}{\psi} \right] \left[ 2s_r s_z + s_z^2 \right] dr \\ M &= \int_a^b \frac{1}{r} \left[ \frac{\beta}{\psi} + \frac{1}{\beta^2} \left( \sqrt{1 - \beta^{2n}} - \frac{\beta}{\psi} \right) \right] \left[ 3s_r s_z + \frac{3}{2} s_z^2 \right] dr \\ Q &= - \int_a^b \frac{1}{r^4} \left[ \frac{-2\beta}{\psi} + \frac{1}{\beta^2} \left( \sqrt{1 - \beta^{2n}} - \frac{\beta}{\psi} \right) \right] \left[ 4s_r^2 + 4s_r s_z + s_z^2 \right] dr. \end{aligned} \quad (16)$$

For the closed end condition we obtain similarly

$$\begin{aligned} 0 &= T\dot{A} + L\dot{C} \\ -\dot{P} &= M\dot{A} + Q\dot{C}. \end{aligned}$$

For the open end condition we can solve (15) for  $\dot{A}$  and  $\dot{C}$  in terms of  $\dot{P}$ , substitute the result into (6) and write, after dividing both sides by  $\dot{P}$

$$\begin{aligned}
\frac{ds_r}{dp} &= \frac{1}{TQ-ML} \left\{ -\frac{\beta}{r^2\psi} \left( \frac{a^3_M}{3} - T \right) \right. \\
&\quad - \frac{1}{r^2\beta^2} \left( \sqrt{1-\beta^{2n}} - \frac{\beta}{\psi} \right) (2s_r^2 + s_r s_z) (a^3_{M-T}) \\
&\quad \left. + \frac{3}{2} s_r s_z \left( L - \frac{a^2_Q}{3} \right) \right\} \\
\frac{ds_z}{dp} &= \frac{1}{KQ-ML} \left\{ \frac{\beta}{\psi} \left( L - \frac{a^2_Q}{3} \right) \right. \\
&\quad - \frac{1}{r^2\beta^2} \left( \sqrt{1-\beta^{2n}} - \frac{\beta}{\psi} \right) (2s_r s_z + s_z^2) \left( \frac{a^3_M}{3} - T \right) \\
&\quad \left. + \frac{3s_z^2}{2} \left( L - \frac{a^2_Q}{3} \right) \right\}. \tag{17}
\end{aligned}$$

For the closed end condition, we obtain similarly

$$\begin{aligned}
\frac{ds_r}{dp} &= \frac{1}{KQ-ML} \left\{ \frac{\beta}{\psi} \frac{T}{r^2} + \frac{1}{\beta^2} \left[ \sqrt{1-\beta^{2n}} - \frac{\beta}{\psi} \right] \left[ \frac{T}{r^2} (2s_r^2 + s_r s_z) + \frac{3L}{2} s_r s_z \right] \right\} \\
\frac{ds_z}{dp} &= \frac{1}{KQ-ML} \left\{ \frac{\beta L}{\psi} + \frac{1}{\beta^2} \left[ \sqrt{1-\beta^{2n}} - \frac{\beta}{\psi} \right] \left[ \frac{T}{r^2} (2s_r s_z + s_z^2) + \frac{3}{2} s_z^2 L \right] \right\} \tag{18}
\end{aligned}$$

The problem of generalized plane strain for either case, open or closed ends, can now be turned into an initial value problem with ordinary differential equations: We discretize the interval  $[a, b]$  by subdividing it into a number of equal intervals. In practice we used 100 intervals. Let the points of subdivision be  $r_i$ ,  $i = 0, \dots, 100$  with  $a = r_0$ ,  $b = r_{100}$ . Let the values of  $s_r$  and  $s_z$  at the mesh points  $r_i$  be  $s_r^i$  and  $s_z^i$



respectively,  $i = 0, \dots, 100$ . We then express all integrals appearing in (16) in terms of the mesh values  $s_r^i, s_z^i$  through a numerical quadrature formula. (In practice, the trapezoidal rule was used.) This allows, then, either the equations (17) (open end) or the equations (18) (closed end) to be replaced by a set of 202 ordinary differential equations for which we take as initial conditions  $s_r^i = s_z^i = 0$  at  $P = 0$ . The boundary and end conditions have been absorbed and the problem is reduced to an initial value problem. Such problems are easily and cheaply solved by digital computers.

Recall now that in the discussion above on the plane strain case (case I), which reduced to the one dimensional problem (9), there was a discussion about how to treat  $\beta$  when it reached unity in order to avoid an error signal when the numerical procedure approximated  $\beta$  by a value greater than one and in order to assure the slight perturbation required to insure a return to an elastic regime at the onset of unloading. For the two cases now under discussion, namely the open and closed end conditions, this instruction has to be generalized to one which will hold when the problem no longer reduces to a non dimensional one. The instruction devised, then, was the following: If  $\beta \geq 1$ , replace  $s_{ij}$  by  $Ns_{ij}/\beta$ , where  $N$  is a number slightly less than unity. During loading, this perturbation should be dying out.

Calculations were carried out on the open end case in order to test the procedure. The results of these calculations left no doubt that the method works well and cheaply. (A typical run on the computer cost between five and ten dollars.) Yield

occurred during loading and a return to elastic behavior occurred at the onset of unloading. Results are shown in figures 1 and 2, in which the ratio of the inner radius to the outer radius was taken to be two. Figure 1 shows  $\beta^2$  versus  $r$  during loading for several values of hydraulic pressure  $P$  (non dimensionalized), which was brought up to a maximum value of 0.74 in steps of 0.01. One can see the yield region ( $\beta = 1$ ) progressing from the inner radius outward for the curves  $P = .70$  and  $P = .74$ . Figure 2 shows the behavior of  $\beta^2$  during unloading: Note a return locally to elastic regimes as  $P$  drops below its maximum. Comparison of figures 1 and 2 shows that unloading behavior is quite different from loading behavior. This is because material points which have experienced yield return to a different elastic regime than they were in before yield.

The program needs refinement before the actual numerical values of stress, in particular residual stress can be reported. A simple Euler method was used,  $N$  was taken to be .99 and the function  $\psi$  was crudely calculated. There is no difficulty at all in making the necessary refinements, and they will be done in order to control error buildup. The project ended before they could be accomplished and so we report the state of affairs at that time.

We conclude now that the essential nature of the procedure has been tested and shown to work. It is easy to handle and economical computation-wise because of the analysis that the unified equations allowed. We feel that it is extremely promising.

We shall end with a brief discussion of the compressible case, for which analysis has been done. The analysis is similar to that for the incompressible case and we shall merely report how it differs from that already presented.

For the compressible case we do not have the equation of incompressibility (4). Instead we have

$$p = -K u_{i,i}, \quad (19)$$

where  $K$  is a constant, namely the bulk modulus. Using (19) and (5), we obtain from the equation of equilibrium (4)

$$\begin{aligned} u = & -\frac{Ar}{2} - \frac{r}{2K} \int_a^r \frac{s_r(\xi) - s_\theta(\xi)}{\xi} d\xi \\ & + \frac{1}{2Kr} \int_a^r \xi s_z(\xi) d\xi - \frac{Pr}{2K} + \frac{C}{r}. \end{aligned} \quad (20)$$

The expression (20) for  $u$  is then substituted into the constitutive equations (1), where again the off diagonal components of  $s_{ij}$  are assumed to vanish. The result is a set of two Volterra integral equations of the second kind for  $\dot{s}_r$  and  $\dot{s}_z$ . Such equations are classical and portend no difficulty in solving for  $\dot{s}_r$  and  $\dot{s}_z$ . The resulting equations then replace (6) and thenceforth the procedure is the same as in the incompressible case.

The general philosophy behind these procedures is that the larger the part of the task relegated to analysis and the smaller the part to computation, the better, more accurate and cheaper will be the computation. In other words, the extra thought that goes into analysis pays off. We are happy that we have been supported in bringing our method to the stage at which it now stands. And



from the evidence accumulated thus far, we are confident that it will prove itself capable of giving excellent results as it is tested on problems and applied to new problems. It should become an excellent tool for problems of autofrettage as well as plasticity in general.

#### References

- [1] Bernstein, B., "A Unified Theory of Elasticity and Plasticity" Int. J. Engineering Science 15, pp. 645-660 (1977).
- [2] Bernstein, B. "Unified Constitutive Relations for Elastic and Plastic Behavior". Recent Advances in Engineering Science - Proceedings of the Fourteenth Annual Meeting, Society of Engineering Science, Lehigh University, Bethlehem, Pennsylvania, Nov. 14-17, 1977, pp. 773-778.
- [3] System/360 Scientific Subroutine Package, Version II, Programmer's Manual H20-0205-1, IBM Company (1967).

#### Figure Captions

- Figure 1. Loading behavior of  $\beta^2 = s_{ij}s_{ij}$  versus radial coordinate  $(r-a)/(b-a)$  for inner radius for  $b/a = 2$  at various values of internal gas pressure  $P$ . This is the incompressible case of generalized plane strain with open end conditions normalized on yield stress so that  $2k^2 = 1$ ,  $n = 20$ .
- Figure 2. Same as figure 1, except that it shows behavior of  $\beta^2$  during unloading.

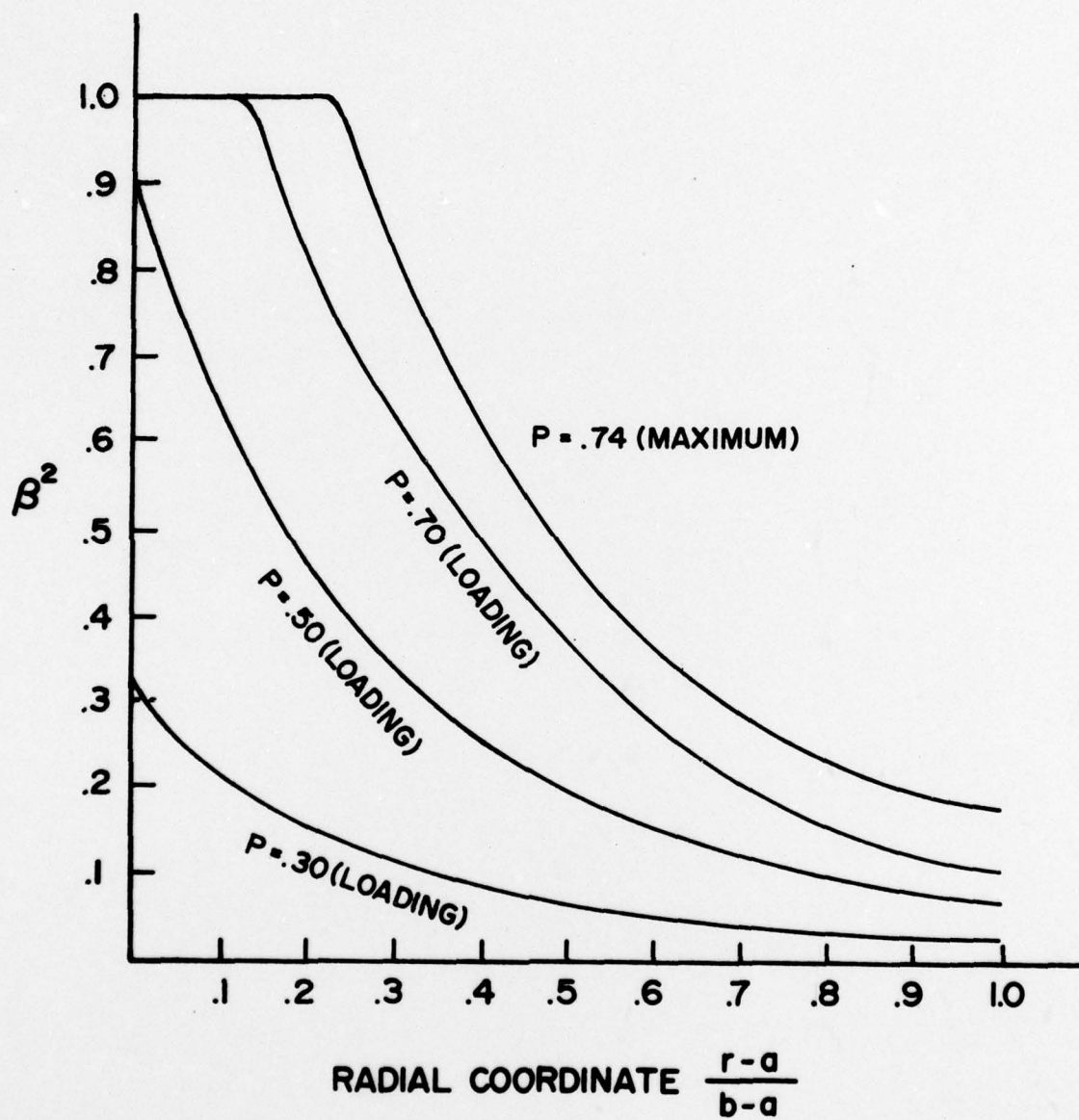


Figure 1

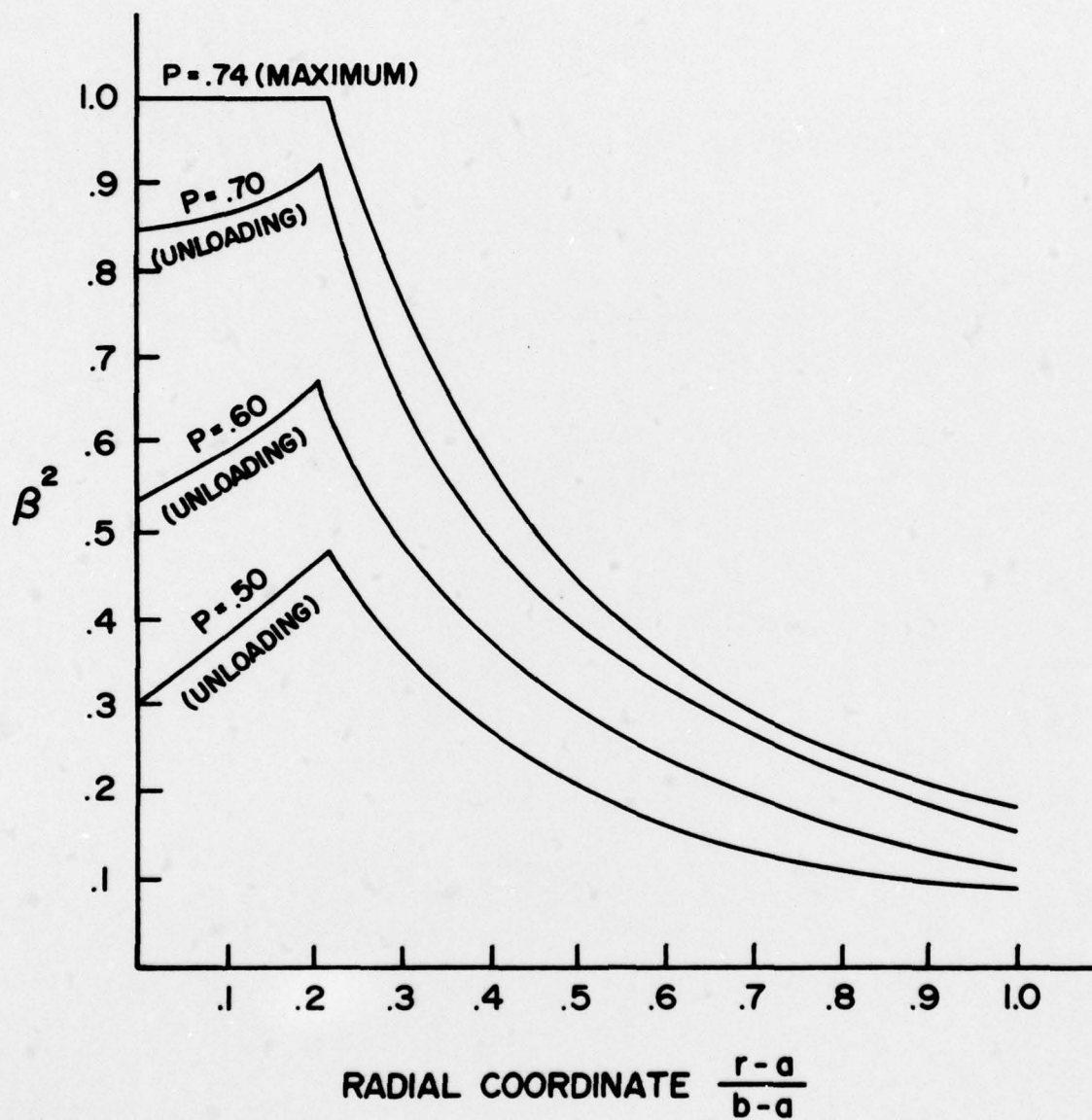


Figure 2